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ABSTRACT

A mathematical procedure, using the method of characteristics, is developed for solving the hydrodynamic flow equations in a nonhomogenous magnetic field for plasma flow along a field line in the presence of a gravitational field. Along a dipole magnetic field line with two supersonic plasma streams symmetrically directed towards the magnetic equator (a situation applicable to the earth's environment) the solution is obtained by assuming the existence of shock discontinuities which propagate along the field line. The solution technique for the time development of the plasma flow is considered for both adiabatic and isothermal flow. A sample calculation for an adiabatic flow state shows explicitly the time evolution of adiabatic cooling and shock heating along a magnetic field line.

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SOLUTION SCHEME FOR TIME DEPENDENT HYDRODYNAMIC PLASMA FLOW ALONG A MAGNETIC FIELD LINE

Introduction

Usually a sharp drop in the ambient electron density occurs with increasing altitude at a geocentric distance of a few earth radii in the earth's equatorial plane. This drop, referred to as the plasmapause, corresponds to the boundary separating plasma that drifts always across closed (i.e., both ends intersect the surface of the earth) magnetic field lines from plasma that at some time in its motion drifts onto polar magnetic field lines which connect the earth to the interplanetary magnetic field (Nishida, 1966). On the latter field lines – the "open" lines – plasma is lost from the earth's environs by transport along the lines of force. This depletion along the open polar field lines occurs in the form of a supersonic polar wind of H^+ ions directed away from the earth (Banks and Holzer, 1968).

Since the plasma which is transported along the open polar field lines also has a drift component perpendicular to the magnetic field lines due to the solar-wind magnetosphere interaction, the supersonic polar wind will convect onto closed magnetic field lines as shown in Figure 1. Assuming symmetry about the magnetic equatorial plane, supersonic streams directed away from the earth in both hemispheres will be convected onto closed magnetic field lines. These streams

will collide near the equator producing collisionless plasma shocks (Banks, et al., 1970) which propagate down the field lines towards the earth.

This paper will describe a mathematical scheme for solving the differential equations which govern the time development of plasma flow along a closed magnetic dipole field line when two supersonic streams directed along the field line are assumed initially to collide at the equator forming plasma shock waves. Although the differential equations to be considered are expressed in terms of a plasma state only, they also describe the flow of a compressible fluid along a channel with varying cross section. Hence the solution scheme to be described also has applicability within the field of fluid dynamics.

System of Equations

Only the plasma flow at high altitudes (i.e., above a few thousand kilometers) where the plasma consists of essentially H^+ ions and electrons is considered for simplicity. Chemical production and loss processes are ignorable in this region. If adiabatic flow is assumed (isothermal flow will be treated later) the hydromagnetic relations (see for example Montgomery and Tidman, 1964) describing plasma flow along the direction of the magnetic field can be written:

$$\rho_t + v\rho_s + \rho v_s = \frac{\rho v}{B} B_s$$

$$\rho v_t + \rho v v_s + P_s = -\rho g \qquad (1)$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial s}\right)(P\rho^{-\gamma}) = 0$$

where an isotropic pressure and a one dimensional coordinate system along the field line is assumed. The first of these equations is the continuity equation for flow in a tube (i.e., a magnetic flux tube) of varying cross section; the second is the momentum equation; and the last corresponds to the energy equation for adiabatic flow. The parameter t denotes time, s is the coordinate distance along the field line under consideration, ρ is the mass density (equal approximately to nM where n is the electron-ion number density and M is the proton mass), v is the flow velocity along the magnetic field direction, B is the magnetic field, P is the total plasma pressure ($P = nkT$ where T is the sum of the electron and ion temperatures and k is Boltzman's constant), g is the component of the gravitational acceleration along the field line and γ is the ratio of specific heats. The subscripts denote partial differentiation.

Since the plasma parameters usually measured by satellite probes are in number densities and plasma temperatures, a more convenient form for these conservation relations is:

$$n_t + vn_s + nv_s = nvF$$

$$nv_t + nv\,v_s + \frac{kT}{M} n_s + \frac{k}{M} nT_s = -ng \qquad (2)$$

$$n(T_t + vT_s) + (1 - \gamma)\, T\,(n_t + vn_s) = 0$$

where F is defined as B_s/B. This system of equations is a set of quasilinear partial differential equations of the first order for the three dependent variables

v, n and T which are functions of the two independent variables t and s. These equations are to be transformed into a system more suitable to a numerical solution.

Characteristic Equations

A system of hyperbolic differential equations may be transformed into a system of total differential equations by choosing appropriate paths in the s-t plane (see Courant and Friedricks, 1948). As is shown in the Appendix, the hydrodynamic equations (2) are hyperbolic provided, as the case in real physical systems, the temperature remains positive. Hence the set of partial differential equations previously developed can be transformed into a system of total differential equations along characteristic directions in the s-t plane.

The characteristic directions for the hydrodynamic equations can be obtained directly by forming linear combinations of these equations and determining which of these combinations involve differentiation of the three dependent variables v, n and T along the same direction ($t(\sigma)$, $s(\sigma)$). For convenience the number density n will be expressed in terms of $\zeta = \ln n$ and each equation will be labeled as

$$L_1 = v_t + vv_s + \frac{k}{M} T \zeta_s + \frac{k}{M} T_s + g = 0 \tag{3}$$

$$L_2 = \zeta_t + v\zeta_s + v_s - vF = 0 \tag{4}$$

$$L_3 = (\zeta_t + v\zeta_s) + \frac{1}{(1-\gamma)T}(T_t + vT_s) = 0. \tag{5}$$

The linear combination can then be written

$$L = \sum_{\mu=1}^{3} \lambda_\mu L_\mu =$$

$$\lambda_1 v_t + (\lambda_1 v + \lambda_2)\, v_s + (\lambda_2 + \lambda_3)\, \zeta_t + \left(\lambda_1 \frac{kT}{M} + \lambda_2 v + \lambda_3 v\right)\zeta_s$$
$$+ \frac{\lambda_3}{(1-\gamma)T}\, T_t + \left(\lambda_1 \frac{k}{M} + \lambda_3 \frac{v}{(1-\gamma)T}\right) T_s + (\lambda_1 g - \lambda_2 vF) = 0. \tag{6}$$

The equations describing the characteristic directions are, by inspection of (6) :

$$s_\sigma = \left(\frac{\lambda_2 + \lambda_1 v}{\lambda_1}\right) t_\sigma \tag{7}$$

$$s_\sigma = \left(\frac{\lambda_1}{\lambda_2 + \lambda_3} \frac{kT}{M} + v\right) t_\sigma \tag{8}$$

$$s_\sigma = \left(\frac{\lambda_1 (1-\gamma)}{\lambda_3} \frac{kT}{M} + v\right) t_\sigma . \tag{9}$$

Solving these equations for the ratios λ_2/λ_1 and λ_3/λ_1 when $\lambda_1 \neq 0$ yields the characteristic directions

$$\frac{s_\sigma}{t_\sigma} = v \pm c \tag{10}$$

where $c \equiv \gamma kT/M$ is defined as the acoustic speed. The third characteristic direction corresponds to $\lambda_1 = \lambda_2 = 0$:

$$\frac{s_\sigma}{s_\sigma} = v. \tag{11}$$

For convenience these characteristic directions will be expressed as

$$\frac{s_\sigma}{t_\sigma} = v + \beta c \tag{12}$$

where $\beta = +1, 0, -1$.

The corresponding linear combinations of the hydrodynamic equations (6) along the characteristic directions become:

$$\beta = +1: \quad v_\sigma + \frac{1}{\gamma} c \, \zeta_\sigma + \frac{2}{\gamma} c_\sigma + (g - cvF) \, t_\sigma = 0 \tag{13}$$

$$\beta = 0 \ : \quad \zeta_\sigma - \frac{2}{(\gamma - 1)c} c_\sigma = 0 \tag{14}$$

$$\beta = -1: \quad v_\sigma - \frac{1}{\gamma} c \, \zeta_\sigma - \frac{2}{\gamma} c_\sigma + (g + cvF) \, t_\sigma = 0 \tag{15}$$

where the subscript σ denotes differentiation along the characteristic. The six characteristic equations – (12) through (15) – replace the original three flow equations (3) through (5).

Solution Scheme and Boundary Conditions

Assume that the variables n, v and T are specified along a given boundary curve (this curve cannot be a characteristic) in s-t space. Consider a point (s, t) near the boundary curve (see Figure 2). If no discontinuities exist at this point, it will correspond to the intersection of three (i.e., $\beta = -1, 0, 1$) distinct characteristics which also intersect the boundary curve. The variation of the variables n, v and T along these characteristic directions will be described by equations (13), (14) and (15).

If the computation point (s, t) is selected very near the boundary curve, then the characteristics will be essentially straight lines between the boundary curve and (s, t) and will be described by $ds/dt = v_\beta + \beta c_\beta$ where $\beta = +1, 0, -1$. The subscripts now denote boundary values and not differentiation (e.g., v_1 is the velocity at the point where the boundary curve is intersected by the $\beta = 1$ characteristic passing through (s, t). Differentiation will be denoted explicitly in the work to follow so this change of notation should cause no difficulties.

For each selected point (s, t) near the boundary, the boundary curve is searched to locate the characteristics which pass through (s, t) and to determine the boundary values (i.e., n_β, v_β, T_β where $\beta = -1, 0, 1$). The finite difference equations corresponding to the characteristic equations (13), (14) and (15) can then be solved at the point (s, t). The finite difference equations can best be solved if considered in matrix form:

$$\begin{bmatrix} 1 & \dfrac{c_1}{\gamma} & \dfrac{2}{\gamma} \\ 0 & 1 & -\dfrac{2}{(\gamma-1)\,c_0} \\ 1 & -\dfrac{c_{-1}}{\gamma} & -\dfrac{2}{\gamma} \end{bmatrix} \begin{bmatrix} v \\ \zeta \\ c \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{16}$$

where

$$b_1 = v_1 + \frac{1}{\gamma} c_1 \zeta_1 + \frac{2}{\gamma} c_1 - (\overline{g}_1 - c_1 v_1 \overline{F}_1)\,\Delta t_1$$

$$b_2 = \zeta_0 - \frac{2}{(\gamma - 1)}$$

$$b_3 = v_{-1} - \frac{c_{-1}\zeta_{-1}}{\gamma} - \frac{2}{\gamma} c_{-1} - (\overline{g}_{-1} + c_{-1} v_{-1} \overline{F}_{-1})\,\Delta t_{-1}$$

and $\overline{g}_\beta$, $\overline{F}_\beta$ denote averages along the segment of the β characteristic between the point (s, t) and the boundary whereas Δt_β is the time increment along this characteristic segment.

After the solutions are generated along a curve located a short distance from the initial boundary (e.g., along the dashed curve in Figure 2), then this curve can be treated as a new boundary curve for further calculations. By proceeding in this manner, the solution can be generated at each point in s-t space at which the three intersecting characteristics can be traced back to the initial boundary curve. It remains now to determine what type of boundary curve is best suitable for the problem under consideration.

Since the plasma state along the earth's outer magnetic field lines will depend upon the rate at which ionization is supplied from the lower ionosphere and upon the rate at which the ionospheric parameters change, a natural choice for a boundary condition is the specification of n, v and T as a function of time at a base level altitude. This base level point (s_0) must be within the protonosphere since the flow equations have been developed assuming a neutral proton-electron plasma. However if the plasma flow is everywhere supersonic along the field line, this boundary condition alone is not sufficient to determine the solution at every point along the field line. This is due to the fact that the only points (at time t) which correspond to the intersection of three distinct characteristics emanating from the boundary curve $s = s_0$ are those points with s coordinates smaller than or equal to the s position (at time t) of the $\beta = -1$ characteristic which passes through the point (s_0, t = 0). As seen in Figure 3, these points (region C) do not cover all of s - t space.

If, on the other hand, boundary conditions are also specified along the boundary t = 0, then the solution can be obtained everywhere in s - t space. This is readily seen in Figure 3, where region A depicts those points at which the solution is obtainable using characteristics which cross the t = 0 boundary only, and where B defines the s - t region in which solutions are generated using a mixture of characteristics, some of which cross the t = 0 boundary and some of which cross the $s = s_0$ boundary curve.

It should be noted that the $t = 0$ and $s = s_0$ boundary values must be selected so that they are compatible with one another. For example if subsonic flow exists near the point $(s_0, t = 0)$, then $\beta = -1$ characteristics (i.e., $ds/dt = v - c$) emanating from the boundary curve segment $t = 0$ near (s_0, o) may cross the curve $s = s_0$. In such a case the value of n, v and T cannot be selected arbitrarily on both boundary curve segments, but must be chosen so that equation (20) is satisfied. Even in the supersonic regime, if the boundary values are not selected properly, accidental discontinuities can be introduced near $(s_0, 0)$.

From the physical viewpoint the most convenient computation scheme consists of determining the plasma state at equally spaced points along the entire field line as a function of time. That is, initially the solution is obtained at all of the selected points along the field line at time Δt, where Δt is a time increment selected small enough that the characteristic segments between the initial boundary $t = 0$ and the computation curve $t = \Delta t$ in (s, t) space are approximately straight lines. The values of v, n and T determined at $t = \Delta t$ are then used to generate the solution along the field line at $t = 2\Delta t$, etc. This procedure will develop explicitly the time evolution of the plasma flow along the field line.

Shock Problem

As indicated in the introduction, the two supersonic plasma streams (one in the northern hemisphere, the other in the southern hemisphere) directed along the chosen field line away from the earth will interact initially at the equator

forming shock discontinuities. Assuming equinoctal conditions (i.e., reflection symmetry of the plasma state about the magnetic equatorial plane) two shocks, one on each side of the equator, will be formed initially at infinitesimal distances from the equatorial plane. The existence of these shocks satisfies the symmetry condition at the equator (i.e., $v = 0$) as the upward directed supersonic flow becomes subsonic in its passage through the shock. After the shocks have been formed initially at the equator, they will propagate along the field line towards the earth as shown in Figure 4. In the following sections only one of the two shocks will be treated explicitly – i.e., the plasma flow will be developed explicitly along only half of the considered field line. The flow with its associated shock on the other half of the field line can be obtained by a simple mirror reflection about the equatorial plane.

Shock Relations

In order to relate the plasma state on the earth side (i.e., upstream) of the shock to the plasma state on the equator side (i.e., downstream) use is to be made of the fact that conservation of the particle number flux, momentum flux, and energy flux must be maintained through the shock. If the plasma parameters on the earth side of the shock discontinuity are denoted by the subscript 0 and the equator side values by the subscript 1, these conservation relations are (see Courant and Friedricks, 1948, for the deviation of these relations from the fluid equations):

$$\rho_1 v_1' = \rho_0 v_0'$$

$$(\rho_1 v_1)v_1' - (\rho_0 v_0)v_0' = P_0 - P_1 \tag{17}$$

$$\rho_1 \left(\frac{1}{2}\ v_1^2 + e_1\right) v_1' - \rho_0 \left(\frac{1}{2}\ v_0^2 + e_0\right)\ v_0' = P_0 v_0 - P_1 v_1$$

where the primes denote flow velocity with respect to the shock which is moving with velocity V_s (e.g., $v_1' = v_1 - V_s$) and where e is the internal energy/mass of the plasma. It should be noted that the velocities are considered to be positive when directed along the field line away from the earth.

In equations (17) if $\gamma \neq 1$ (the isothermal case $\gamma = 1$ will be discussed in a later section) the energy density e can be expressed as

$$e = \frac{1}{(\gamma - 1)}\ \frac{kT}{M} = \frac{1}{\gamma(\gamma - 1)}\ c^2 \tag{18}$$

whereas the pressure P, as defined earlier, is just nkT and the mass density ρ is nM. Using these expressions, equations (17) are expressible in terms of n and c rather than ρ, e and P:

$$n_1 v_1' = n_0 v_0'$$

$$n_1 v_1 v_1' - n_0 v_0 v_0' = \frac{1}{\gamma}\ (n_0 c_0^2 - n_1 c_1^2) \tag{19}$$

$$n_1 v_1' \left(\frac{1}{2}\ v_1^2 + \frac{1}{\gamma(\gamma - 1)}\ c_1^2\right) - n_0 v_0' \left(\frac{1}{2}\ v_0^2 + \frac{1}{\gamma(\gamma - 1)}\ c_0^2\right) = \frac{1}{\gamma}\ (n_0 v_0 c_0^2 - n_1 v_1 c_1^2).$$

When upward directed supersonic flow exists everywhere downstream of the shock, the solution to the flow equations on the earth side can be determined completely at all points using the previously discussed procedure. The shock discontinuity does not enter into these computations except as an upper cutoff boundary. However the plasma state on the equator side of the shock cannot be determined without considering the effects of the shock since all of the characteristics connecting this region to the $s = s_0$ boundary would pass through the shock. In fact, as will be seen, the characteristics on the equator side of the shock are determined in part by using the shock path in $s - t$ space as a boundary curve. The corresponding boundary values on the equator side of the shock can be determined by solving the shock jump conditions – equations (19).

Since the earth side flow can be determined without taking into account the shock whereas the equator side flow cannot, equations (17) are best solved for the parameters on the equator side of the shock in terms of the parameters upstream of the shock. Such a solution is given by the following expressions:

$$\frac{n_0}{n_1} = \frac{(\gamma - 1)\, v_0'^2 + 2c_0^2}{(\gamma + 1)\, v_0'^2} \tag{20}$$

$$\frac{v_1'}{v_0'} = \frac{n_0}{n_1} \tag{21}$$

$$c_1^2 = \left(\frac{n_0}{n_1}\right) [c_0^2 - \gamma v_0' (v_1' - v_0')] \tag{22}$$

These equations correspond to the Rankine Hugoniot equations.

Equations (20) through (22) determine the downstream parameters at the shock only when the shock velocity or one other downstream parameter is known. Since a nonsteady state is under consideration, these equations will not, in general, form a closed system except at t = 0 when the shock is only an infinitesimal distance from the equator and $v_1 = 0$ by symmetry. In such a case equation (21) can be replaced by the relation for the shock velocity which is:

$$V_s = -\alpha - \sqrt{\alpha^2 + \beta^2} \tag{23}$$

where

$$\alpha = \frac{1}{4}(\gamma - 3)v_0$$

and

$$\beta = \frac{1}{2}(\gamma - 1)\, v_0^2 + 2c_0^2 \, .$$

When $t \neq 0$ the downstream flow equations must be utilized to determine the conditions at the shock. This procedure will be discussed in the next section.

Shock Solution

Since the flow on the equator side of the shock depends upon the shock conditions and since the shock jump conditions do not in general form a closed set of equations, another equation must be set up, downstream of the shock, which does not increase the number of dependent variables. One of the previously developed characteristic relations – equation (15) – describing the flow downstream of the shock will be used for this purpose.

Consider the situation near the shock as depicted in Figure 5. The flow at all points on the earth side of the shock, as demonstrated previously, can be determined without reference to the shock. Now assume that the solution has been obtained at all points on the field line downstream of the shock at time t_c. Then since the flow is subsonic at these locations, $\beta = -1$ characteristics connect downstream points at $t = t_c$ to the shock path between t_c and $t_c + \Delta t$ (see Figure 5). Hence equation (15) des ribing variations along these characteristics can be used to complete the solution to the jump conditions. The $\beta = +1, 0$ characteristics, on the other hand, have positive slopes in s - t space and do not connect the $t = t_c$ downstream points to the shock and hence do not directly relate to the solution for the shock motion.

Assume that the shock position at time t_c, denoted by X_0, and the corresponding shock velocity are known (i.e., they were previously computed). Then if the shock velocity changes only slightly in the time interval Δt, the shock velocity at $t = t_c + \Delta t$ can, to a good approximation, be expressed as a linear function of X_1 which denotes the position (as yet unknown) of the shock at time $t_c + \Delta t$. That is:

$$V_s(X_1) = \frac{2(X_1 - X_0)}{\Delta t} - V_s(X_0) \qquad (24)$$

Since the flow solution is known everywhere on the earth side of the shock, the variables n_0, v_0, c_0 are determinable functions of s and hence of X_1 at time $t_c + \Delta t$. Substituting equation (24) into the shock relations (20) through (22) will then yield three simultaneous equations for the four unknowns n_1, v_1, c_1 and X_1.

The dependent variables along the field line on the equator side of the shock at $t = t_c$ are assumed to have been previously determined and are thus known functions of the distance along the field line – in this region these variables are denoted by the subscript I. Therefore the $\beta = -1$ characteristic passing through the shock point $(X_1, t_c + \Delta t)$ relates the downstream shock parameters n_1, v_1, c_1 and the shock position X_1 to the position s_I at which this characteristic crosses the $t = t_c$ line:

$$s_I = X_1 - \left(\frac{v_1 + v_I - c_1 - c_I}{2}\right)\Delta t \tag{25}$$

where v_I, c_I are functions of s_I. The finite difference form of equation (15) along this characteristic segment is:

$$v_1 - v_I - \frac{1}{\gamma}\frac{c_1 + c_I}{n_1 + n_I}(n_1 - n_I) - \frac{2}{\gamma}(c_1 - c_I) + \left(\overline{g} + \frac{(c_1 + c_I)(v_1 + v_I)}{2}\overline{F}\right)\Delta t = 0. \tag{26}$$

This equation in conjunction with equation (25) yields a relation between v_1, n_1, c_1 and the shock position X_1. Hence these relations and the shock jump conditions form a closed set of equations which can be solved for the shock position and the downstream shock variables.

Downstream Solution

The solution at points between the shock and the equator can readily be obtained using the method of characteristics described earlier. However, the boundary curve is not the same as was used for the upstream solution. The $t = t_c$ line segment in $s - t$ space on the equator side of the shock is still a valid boundary curve segment for computing the solution at a later time $t = t_c + \Delta t$. However the $s = s_0$ boundary curve segment used previously is not directly applicable to the downstream solution. It is replaced by the shock path in $s - t$ space. Hence the shock position at time $t_c + \Delta t$ must be computed along with the equatorward shock variables before the solution can be obtained downstream from the shock. The variation of n_1, v_1, c_1 along the shock path is obtained by interpolation between the values at t_c and $t_c + \Delta t$.

Isothermal Flow

The flow solution thus far has been developed explicitly for the case $\gamma \neq 1$. If an isothermal state ($\gamma = 1$) exists along the field line (ignoring the shock momentarily) the solution scheme must be modified slightly. When $\gamma = 1$ the temperature, which is specified as a function of time at the base level s_0 (i.e., along the $s = s_0$ boundary curve segment), is known at all points along the field line and only the $\beta = +1$ and -1 characteristic equations are required for a solution. The energy equation (14) is no longer applicable nor is it needed for a solution. Otherwise the solution is obtained in the manner previously described.

When the shock discontinuities are included and isothermal conditions prevail upstream and downstream of the shocks, the field line is separated into two distinct isothermal regions each with its own characteristic temperature. The spatially invariant temperature on the equator side of the shock is coupled to the upstream temperature via the shock jump conditions. The shock process, however, is not isothermal and hence cannot be described by simply setting $\gamma = 1$ in the shock relations (19). The conservation conditions (17) are still valid when the temperature is spatially invariant on both sides of the shock, and the first part of (18) relating the internal energy to the temperature is correct if γ is taken to be the value leading to the internal energy per mass (e) of the plasma (e.g., if three internal degrees of freedom exist $\gamma = 5/3$). However the relationship between e and the sound speed c must be modified since the isothermal sound speed on either side of the shock is $c^2 = kT/M$ and hence is not related to the factor γ describing the energy transfer through the shock in the first equality of equation (18). That is, (18) is replaced by

$$e = \frac{1}{(\gamma - 1)} c^2$$

where $\gamma \neq 1$. When this change is made, the modified form of the jump conditions (20) through (22) are readily obtainable and the solution scheme then follows the procedure described previously using the $\beta = +1$ and -1 characteristics.

Sample Computation

A sample calculation will show explicitly the type of time evolution which is sought. The solution is developed along a field line of the earth's magnetic field (assumed dipolar) which intersects the equatorial plane at a geocentric distance of 6 earth radii (1 earth radius corresponds to 6370 kilometers). For simplicity the dependent variables n, v and T are fixed at constant values along the boundary curves (i.e., along the field line initially, and at all times at the base level-selected to be at an altitude of 3000 km). The fixed values selected were:

$$n = 300 \text{ particles/cc}$$

$$v = 11 \text{ km/sec}$$

$$T = 8000°K$$

These values were selected arbitrarily and are not to be taken as representative of any real geophysical state.

With these boundary conditions the flow develops along the field line as shown in Figure 6. The shock which is formed initially at the equator is accelerated towards the base of the field line leaving behind a hot, relatively dense, subsonic flow. In this case a steepening compression wave is seen to propagate towards the shock. The computed density and temperature on the earth side of the shock in a short period of time approach a state of rapid decrease with increasing distance along the field line. This results from the adiabatic expansion of the plasma along the diverging magnetic flux tube. These computations indicate the

type of flow behavior which can be explored using the solution scheme discussed in this paper.

APPENDIX A

The equations under consideration can be expressed in the general form

$$L_\mu = a_{\mu\sigma}^{(\nu)} u_{x_\nu}^{(\sigma)} + f_\mu = 0 \tag{A-1}$$

where the summation convention is assumed (σ is summed over the number of dependent variables, ν is summed over the number of independent variables). The greek letters denote labeling indices whereas the subscript x_ν denotes differentiation with respect to the independent coordinate x_ν. The coefficients $a_{\mu\sigma}$ are, in general, functions of the independent variable x_ν and the dependent variable $u^{(\sigma)}$.

In equations (A-1) the functions $u^{(\sigma)}$ are differentiated in different directions. It would simplify the numerical computations if characteristic surfaces could be determined for which a linear combination $L = \lambda_\mu L_\mu$ of these equations involves derivatives of the functions $u^{(\sigma)}$ only in the direction of the surface element. The condition for the existence of the multipliers λ_μ is the determinant equation

$$\| a_{\mu\sigma}^{(\nu)} \zeta^{(\nu)} \| = 0 \tag{A-2}$$

for the components of the normal vector $\{\zeta^{(\nu)}\}$ characterizing the surface elements (see Courant and Friedricks, 1948).

The system of equations (A-1) is identical to equations (3), (4) and (5) if the variables are defined as:

$$\mu = 1, 2, 3; \; \sigma = 1, 2, 3; \; \nu = 1, 2$$

$$\begin{aligned} u^{(1)} &= v(s, t) & f_1 &= ng \\ u^{(2)} &= n(s, f) & f_2 &= -nvF \\ u^{(3)} &= T(s, t) & f_3 &= 0 \end{aligned} \qquad \text{(A-3)}$$

and the matrix elements $a_{\mu\sigma}^{(\nu)}$ where μ denotes the row and σ the column are given by

$$a_{\mu\sigma}^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ u^{(2)} & 0 & 0 \\ 0 & (1-\gamma)u^{(3)} & u^{(2)} \end{bmatrix}$$

(A-4)

and

$$a_{\mu\sigma}^{(2)} = \begin{bmatrix} u^{(2)} & u^{(1)} & 0 \\ u^{(2)}u^{(1)} & \frac{k}{M} u^{(3)} & \frac{k}{M} u^{(2)} \\ 0 & (1-\gamma)u^{(3)}u^{(1)} & u^{(2)}u^{(1)} \end{bmatrix}$$

Substituting these relations into equation (A-2) and expanding the determinant yields the relation

$$(u^{(2)})^2 \, (\eta + u^{(1)}) \left[\gamma \frac{k}{M} u^{(3)} - \eta^2 - 2u^{(1)} \eta + (u^{(1)})^2\right] = 0 \qquad \text{(A-5)}$$

where

$$\eta \equiv \frac{\zeta^{(1)}}{\zeta^{(2)}}$$

defines the direction of the vector $\{\zeta^{(\nu)}\}$.

One solution of (A-5) is readily seen to be $\eta = -u^{(1)}$ which is always real. The other solutions are obtained from the quadratic relation formed when the square bracketed term in (A-5) is set to zero. Both solutions of the quadratic equation in η are real if $\gamma k/M\ u^{(3)} > 0$. Since $u^{(3)} \equiv T$ and since the temperature is always positive in a real physical state, this latter relation is always satisfied. Hence three distinct characteristic surface elements exist and the system of hydrodynamic equations under consideration are totally hyperbolic (Courant and Hilbert, 1937), making a numerical solution along characteristic directions possible.

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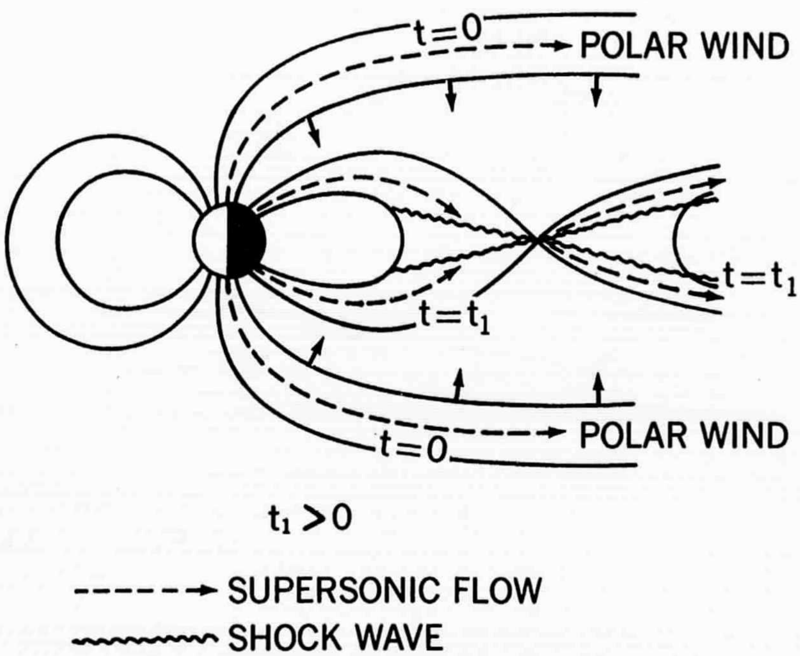

Figure 1. Flow Character of H^+ Ions Within the Earth's Magnetic Field.

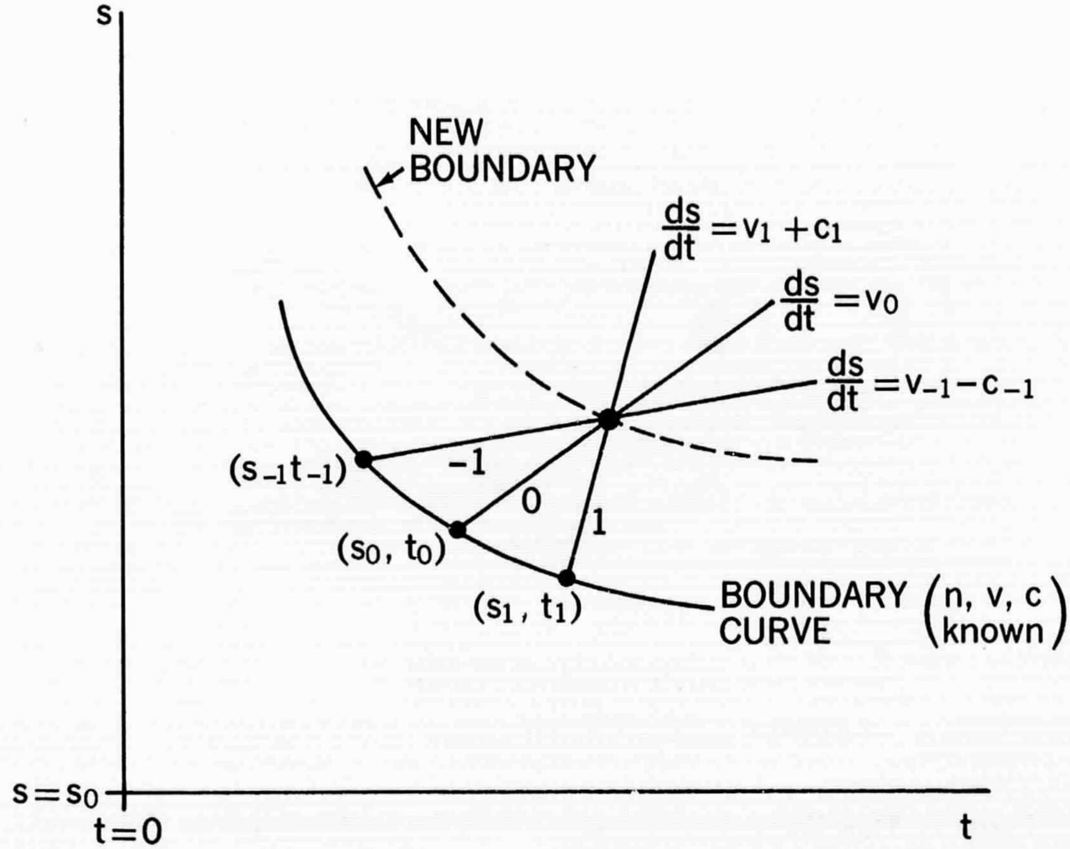

Figure 2. This Diagram Illustrates How a Computational Point Is Connected to a Boundary Curve via Characteristics.

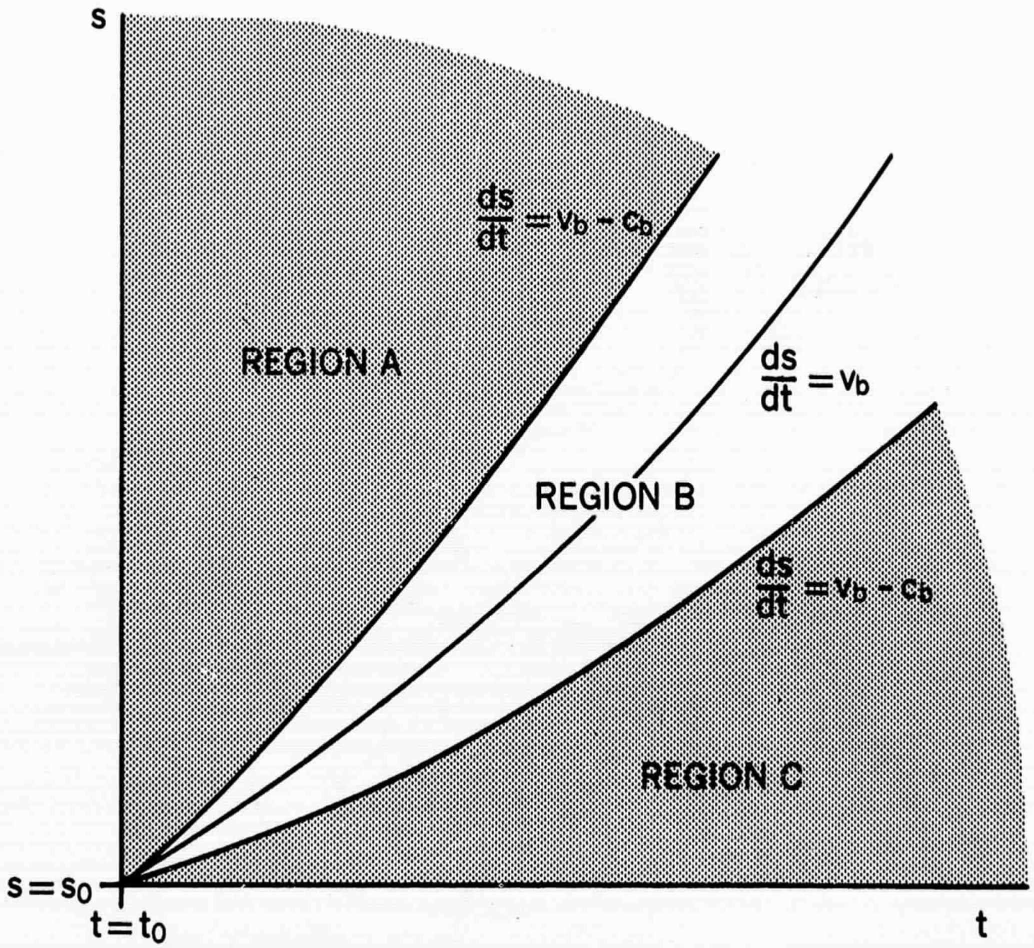

Figure 3. Region A Is Covered by Characteristics Emerging from the $t = t_0$ Boundary, Region B Is Connected to both the $t = t_0$ and $s = s_0$ Boundary Curve Segments, Whereas Region C Is Coupled to Only the $s = s_0$ Boundary. The Subscript b Denotes Values Along the Characteristic Emerging from (s_0, t_0).

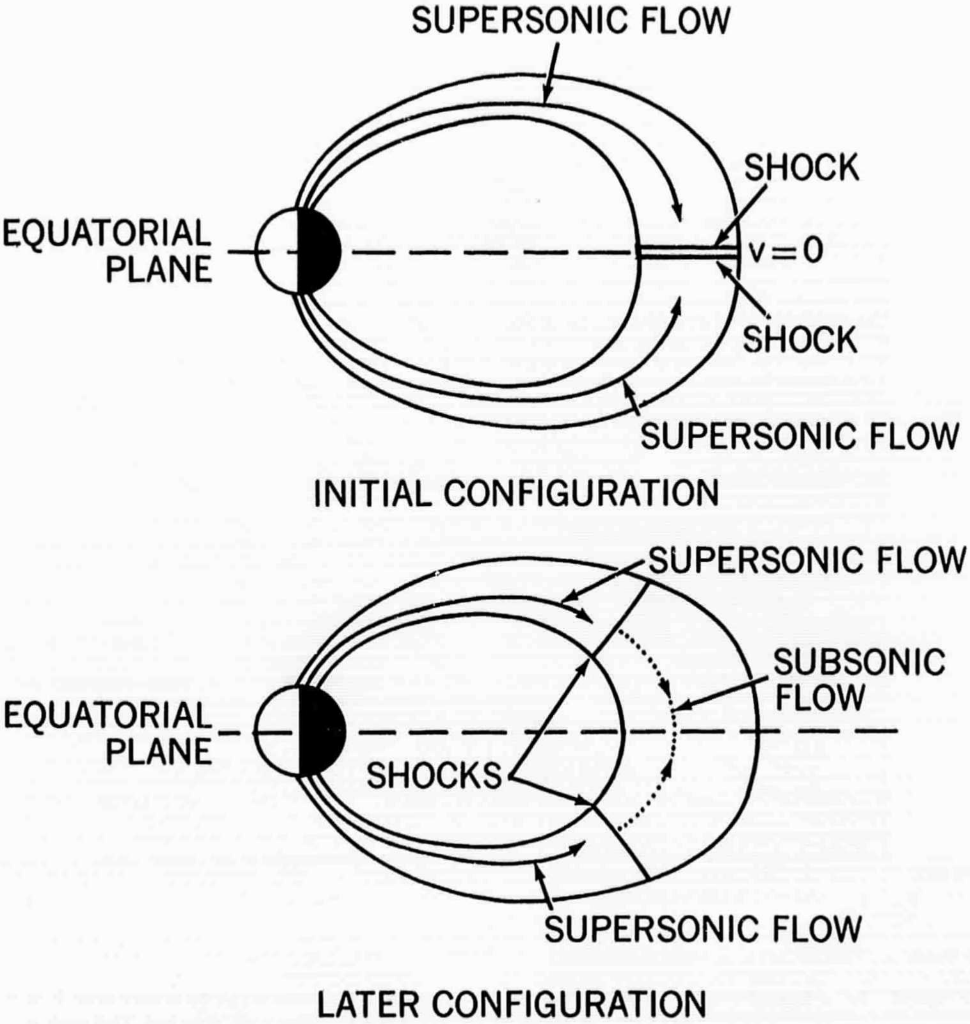

Figure 4. The Shocks Produced Initially at the Equator Propagate Towards the Earth Leaving Behind a Subsonic Flow.

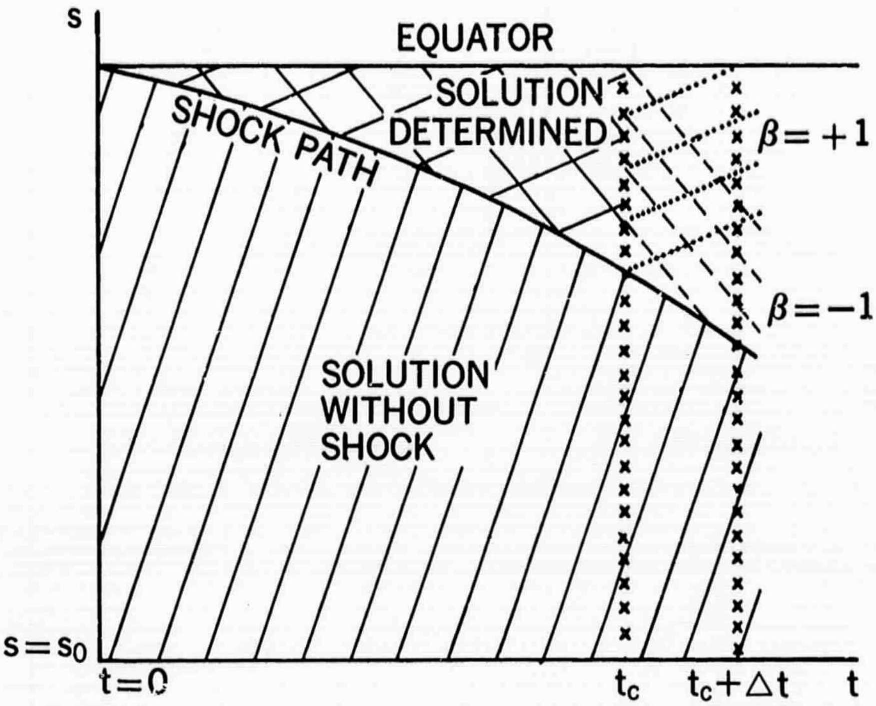

Figure 5. The Figure Shows That $\beta = -1$ Characteristics (dashed lines) Connect Computed Points (x's) to the Shock Path Whereas the $\beta = +1$ Characteristics (dotted curves) on the Equator Side of the Shock do not.

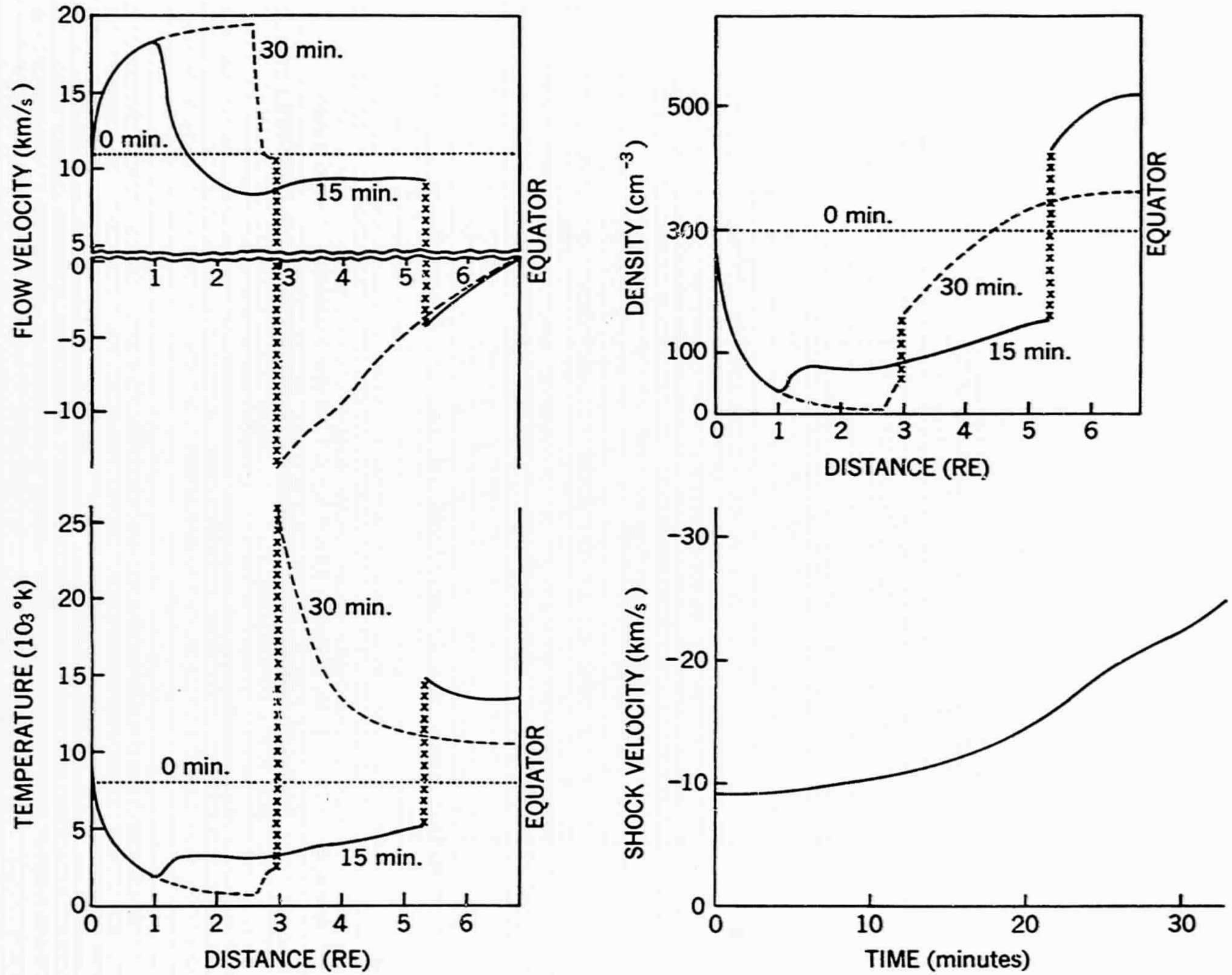

Figure 6. Flow Development Along a Dipole Field Line Which Crosses the Earth's Equatorial Plane at 6 Earth Radii with Constant Dependent Variables Along the Boundary Curves